SOLUTIONS TO SELECTED EXERCISES IN

A GENTLE INTRODUCTION TO OPTIMIZATION

Chapter 1.2 – Linear programs (pp. 9–14)

Exercise 1 Let x_1 (resp. $x_2, \ldots, x_5$) denote the number of serving of carrots (resp. potatoes, bread, cheese, peanut) to buy. Let $x = (x_1, x_2, x_3, x_4, x_5)^\top$. The required LP is as follows,

$$
\begin{array}{ll}
\min & (0.14, 0.12, 0.2, 0.75, 0.15)x \\
\text{subject to} & \\
& \begin{pmatrix} 23 & 171 & 65 & 112 & 188 \\ 0.1 & 0.2 & 0 & 9.3 & 16 \\ 0.6 & 3.7 & 2.2 & 7 & 7.7 \\ 6 & 30 & 13 & 0 & 2 \end{pmatrix} x \geq \begin{pmatrix} 2000 \\ 50 \\ 100 \\ 250 \end{pmatrix} \\
& x \geq \mathbb{0}
\end{array}
$$

Exercise 2

(a) Let x_i be the number of items of type i to produce, where $i = 1$ for skim milk, 2 for 2% milk, 3 for whole milk, 4 for table cream and 5 for whipping cream. One constraint is obvious: the total volume should be equal to the total produced, which is 750 litres. The other constraint is that the milk-fat percentages should be correct. The total amount of milk-fat is 0.037 times 500 plus 0.049 times 250, which is 30.75. x_1 litres of skimmed milk produce $0.001x_1$ litres of milk-fat, and so on. Then of course there are non-negativity constraints. Let $x = (x_1, x_2, x_3, x_4, x_5)^\top$. Then the formulation is,

$$
\begin{array}{ll}
\max & (0.1, 0.15, 0.2, 0.5, 1.2)x \\
\text{subject to} & \\
& \begin{pmatrix} 2 & 2 & 2 & 0.6 & 0.3 \\ 0.001 & 0.02 & 0.04 & 0.15 & 0.45 \end{pmatrix} x = \begin{pmatrix} 750 \\ 30.75 \end{pmatrix} \\
& x \geq \mathbb{0}
\end{array}
$$

(b) If p is the proportion of milk-fat in skim milk, then the new rules say, $0 \leq p \leq 0.001$, And we must have,

$$
(\star) \qquad px_1 + 0.02x_2 + 0.04x_3 + 0.15x_4 + 0.45x_5 = 30.75.
$$

However, this is not a linear constraint, so it cannot be used directly in an LP. ($\star$) implies that,

$$0.02x +_2 +0.04x_3 + 0.15x_4 + 0.45x_5 \leq 30.75$$
$$0.001x_2 + 0.02x_2 + 0.04x_3 + 0.15x_4 + 0.45x_5 \geq 30.75.$$

Conversely, if these two conditions hold, then ($\star$) holds for some p satisfying $0 \leq p \leq 0.001$. So the new formulation is

$$\begin{aligned} \max \quad & (20.1, 0.15, 0.2, 0.5, 1.2)x \\ \text{subject to} \quad & \begin{pmatrix} 2 & 2 & 2 & 06 & 0.3 \\ 0 & 0.02 & 0.04 & 0.15 & 0.45 \\ 0.001 & 0.02 & 0.04 & 0.15 & 0.45 \end{pmatrix} x \begin{matrix} = \\ \leq \\ \geq \end{matrix} \begin{pmatrix} 750 \\ 30.75 \\ 30.75 \end{pmatrix} \\ & x \geq \mathbb{0} \end{aligned}$$

Exercise 3

(a) We will introduce a variable x_i for

$$i \in \{\text{Tom, Peter, Nina, Samir, Linda, Gary, Bob}\}$$

representing the salary of i. We create one constraint for each of the constraints.

- *Tom wants at least $20,000 or he will quit;*

(1) $$x_{\text{Tom}} \geq 20000$$

- *Peter, Nina, and Samir each want to be paid at least $5000 more than Tom;*
  We need a constraint for each of the three employees:

(2) $$x_{\text{Peter}} \geq x_{\text{Tom}} + 5000$$
(3) $$x_{\text{Nina}} \geq x_{\text{Tom}} + 5000$$
(4) $$x_{\text{Samir}} \geq x_{\text{Tom}} + 5000$$

- *Gary wants his salary to be at least as high as the combined salary of Tom and Peter;*

(5) $$x_{\text{Gary}} \geq x_{\text{Tom}} + x_{\text{Peter}}$$

- *Linda wants her salary to be $200 more than Gary;*
  This is interpreted as Linda wanting *exactly* $200 more than Gary:

(6) $$x_{\text{Linda}} = x_{\text{Gary}} + 200$$

- *The combined salary of Nina and Samir should be at least twice the combined salary of Tom and Peter;*

(7) $$x_{\text{Nina}} + x_{\text{Samir}} \geq 2(x_{\text{Tom}} + x_{\text{Peter}})$$

- *Bob's salary is at least has high as that of Peter and at least as high as that of Samir;*
  Once more, we need two constraints here.

(8) $$x_{\text{Bob}} \geq x_{\text{Peter}}$$
(9) $$x_{\text{Bob}} \geq x_{\text{Samir}}$$

- *The combined salary of Bob and Peter should be at least \$60,000;*

(10) $$x_{\text{Bob}} + x_{\text{Peter}} \geq 60000$$

- *Linda should make less money than the combined salary of Bob and Tom.*
  There is no way that we can express this constrained linearly as we are not allowed to use strict inequalities, and since all variables are fractional (as opposed to integral). We interpret this constraint as "Linda's salary is no more than the combined salary of Bob and Tom".

(11) $$x_{\text{Linda}} \leq x_{\text{Bob}} + x_{\text{Tom}}$$

The entire LP now becomes:

$$\begin{array}{ll} \min & x_{\text{Tom}} + x_{\text{Peter}} + x_{\text{Nina}} + x_{\text{Samir}} + x_{\text{Linda}} + x_{\text{Gary}} + x_{\text{Bob}} \\ \text{subject to} & \\ & (1) - (11) \\ & x \geq \mathbb{0} \end{array}$$

(b) We introduce one new variable M for the maximum salary of any employee. We now add the following constraints to the LP from (a):

(12) $$x_i \leq M \ (i \in \{\text{Tom, Peter, Nina, Samir, Linda, Gary, Bob}\})$$

We also change the objective function to

$$\min \ M,$$

and hence the new LP becomes

$$\begin{array}{ll} \min & M \\ \text{subject to} & \\ & (1) - (12) \\ & x \geq \mathbb{0} \end{array}$$

Suppose that (x, M) is an optimal solution to the above LP. Then one of the constraints in (12) must be satisfied with equality. *Why?*

(c) The objective function in (a) enforces the sum of all salaries to be small; i.e., in other words, the average salary will be small. On the other hand, in solutions to (b) that maximum salary will be small, but the sum might still be large!

Exercise 4 Let x_B (resp. x_F, x_E, x_P, x_D, x_L) be the start of job B (resp. F, E, P, D and L). Let y denote the completion time of the last job. We will want to minimize y. The completion time is going to be after job B is completed. Since it takes 3 weeks to complete job B we have $y \geq x_B + 3$. We have similar constraints for each tasks. Task F can only be started when task B is completed.